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Abstract

We consider free atoms and ions in $\mathbb{R}^3$ interacting with the quantized electromagnetic field. Because of the translation invariance we consider the reduced hamiltonian associated with the total momentum. After introducing an ultraviolet cutoff we prove that the reduced hamiltonian for atoms has a ground state if the coupling constant and the total momentum are sufficiently small. In the case of ions an extra infrared regularization is needed. We also consider the case of the hydrogen atom in a constant magnetic field. Finally we determine the absolutely continuous spectrum of the reduced hamiltonian.

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Contents

1 Introduction.

In [8] V. Bach, J. Fröhlich and M. Sigal consider atoms and molecules with fixed nuclei interacting with the quantized electromagnetic field. If the interaction between the electrons and the quantized radiation field is turned off, the atom or molecule is assumed to have at least one bound state. After introducing an ultraviolet cutoff for the quantized field, they prove that the interacting system has a ground state corresponding to the bottom of the energy spectrum for sufficiently small values of the fine structure constant. In [19] and [25], M. Griesemer, E. Lieb and M. Loss have been able to get rid of the smallness condition concerning the fine structure constant (see also [9]).

In this paper we consider free atoms and ions in $\mathbb{R}^3$ interacting with the quantized electromagnetic field. Because of the translation invariance in $\mathbb{R}^3$ we consider the reduced hamiltonian associated with the total momentum for particles and field. After introducing and ultraviolet cutoff for the quantized electromagnetic field we prove that the reduced hamiltonian for atoms has a ground state if the coupling constant of the interaction between the particles and the field together with the total momentum are sufficiently small. In the case of ions we need to introduce an infrared regularization in order to get the same result. The infrared regularization is not needed in the case of atoms because we are able to use a simple form of the Power-Zienau-Woolley transformation for a moving neutral system of charges (see [8], ([19], [20]). We also consider a nonrelativistic hydrogen atom interacting with a constant magnetic field and with a quantized electromagnetic field. Again a Power-Zienau-Woolley transformation applies and we prove that the reduced Hamiltonian associated with the total momentum has a ground state under the same conditions as without a constant magnetic field. This result has to be compared with [2] where we consider an electron in a classical magnetic field pointing along the x_3-axis interacting with a quantized electromagnetic field. In this case we need an infrared regularization because of the the system is charged.

In the case of one free particle a similar problem has been studied in [10] where T. Chen considers a freely propagating relativistic spinless charged particle interacting with the quantized electromagnetic field. The one-particle sector of Nelson's model has been studied first by J. Fröhlich (see[14], [15]) and more recently by A. Pizzo (see [28],[29]) and J.S. Møller (see [27]).

As in [2] our proof combines the approach associated with hamiltonian which are invariant by translation (see[14], [28], [29], [16]) together with that concerning confined system of charges (see[9], [23], [21], [22], [24], [19], [25], [17]).

Let us finally remark that the same result holds in the case of atoms moving in a waveguide and interacting with the quantized electromagnetic field in the wave guide (see [1]) and in the case of dressed atoms moving in a periodic potential (see [11]).

2 Definition of the model and self-adjointness.

2.1 The Hamiltonian.

We consider N electrons in $\mathbb{R}^3$ of charge e and mass m interacting with a nucleus of charge Ze, mass $m_{\rm ncl}$ and spin S and with photons. We suppose $N \geq 2$. The associated Pauli Hamiltonian in Coulomb gauge is formally given by

$$\begin{aligned} H \equiv H_{N,Z} = & \sum_{j=1}^{N} \frac{1}{2m} \left(p_j - eA(x_j)\right)^2 + \frac{1}{2m_{\rm ncl}} \left(p_{N+1} + ZeA(x_{N+1})\right)^2 \\ & + V_{cl}(x) \otimes 1 + 1 \otimes H_{\rm ph} - \frac{e}{2m} \sum_{j=1}^{N} \sigma_j \cdot B(x_j) + g_{\rm ncl} \frac{Ze}{2m_{\rm ncl}} S \cdot B(x_{N+1}) \end{aligned} \tag{2.1}$$

with

$$V_{cl}(x) \equiv V_{cl}(x_1, \ldots, x_{N+1}) := \sum_{j=1}^{N} \frac{Ze^2}{|x_j - x_{N+1}|} + \sum_{1 \leq i < j \leq N} \frac{e^2}{|x_i - x_j|} \ . \tag{2.2}$$

Here the units are such that $\hbar = c = 1$, $p_j = -i\nabla_{x_j}$, $x_j = (x_{j\,1}, x_{j\,2}, x_{j\,3})$, $\sigma_j = (\sigma_{j\,1}, \sigma_{j\,2}, \sigma_{j\,3})$is the 3-component vector of the Pauli spin matrices for the jth electron. $S = (S_1, S_2, S_3)$ is a $3-$component vector of spin hermitian matrices in $\mathbb{C}^d$ for the nucleus and $g_{\rm ncl}$ is the Landé factor of the nucleus.

The quantized electromagnetic field is formally given by

$$A(x) = \frac{1}{2\pi} \sum_{\mu=1,2} \int d^3k \left(\frac{1}{|k|^{1/2}} \epsilon_\mu(k) e^{-ik\cdot x} a_\mu^\star(k) + \frac{1}{|k|^{1/2}} \epsilon_\mu(k) e^{ik\cdot x} a_\mu(k) \right) \ , \tag{2.3}$$

$$\begin{aligned} B(x) = \frac{i}{2\pi} \sum_{\mu=1,2} \int d^3k \Big\{ & -|k|^{1/2} \left(\frac{k}{|k|} \wedge \epsilon_\mu(k) \right) e^{-ik\cdot x} a_\mu^\star(k) \\ & + |k|^{1/2} \left(\frac{k}{|k|} \wedge \epsilon_\mu(k) \right) e^{ik\cdot x} a_\mu(k) \Big\} \end{aligned} \tag{2.4}$$

where $\epsilon_\mu(k)$ are real polarization vectors satisfying $\epsilon_\mu(k) \cdot \epsilon_{\mu'}(k) = \delta_{\mu\mu'}$, $k \cdot \epsilon_\mu(k) = 0$; $a_\mu(k)$ and $a_\mu^\star(k)$ are the usual annihilation and creation operators acting in the Fock space

$$\mathcal{F} := \oplus_{n=0}^{\infty} L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^n}$$

where $L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^0} = \mathbb{C}$ and $L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^n}$ is the symmetric n-tensor power of $L^2(\mathbb{R}^3, \mathbb{C}^2)$ appropriate for Bose-Einstein statistics. The annihilation and creation operators obey the canonical commutation relations ($a^\sharp = a^\star$ or a)

$$[a_\mu^\sharp(k), a_{\mu'}^\sharp(k')] = 0 \quad \text{et} \quad [a_\mu(k), a_{\mu'}^\star(k')] = \delta_{\mu\mu'} \delta(k - k') \ . \tag{2.5}$$

Finally the Hamiltonian for the photons in the Coulomb gauge is given by

$$H_{\mathrm{ph}} = \sum_{\mu=1,2} \int |k| a^\star_\mu(k) a_\mu(k) d^3k \ . \tag{2.6}$$

The Hilbert space associated with $H_{N,Z}$ is then

$$\mathcal{H} = \mathcal{A}_N \left[L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes^N} \right] \otimes L^2(\mathbb{R}^3, \mathbb{C}^d) \otimes \mathcal{F}.$$

Here $\mathcal{A}_N$ is the orthogonal projection onto the subspace of totally antisymmetric wave functions, as required by the Pauli principle.

As it stands, the Hamiltonian H cannot be defined as a self-adjoint operator in $\mathcal{H}$ and we need to introduce cutoff functions, both in $A(x)$ and in $B(x)$, which will satisfy appropriate hypothesis in order to get a self-adjoint operator in H.

This operator, still denoted by H, commutes with each component of the total momentum of the system denoted by P. We have $P = \left(\sum_{j=1}^{N+1} p_j \right) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k)$ where, for every i, $\mathrm{d}\Gamma(k_i)$ is the second quantized operator associated to the multiplication operator by k_i in $L^2(\mathbb{R}^3, \mathbb{C}^2)$. The joint spectrum of (P_1, P_2, P_3) is the real line. It turns out that H admits a decomposition over the joint spectrum of (P_1, P_2, P_3) as a direct integral :

$$H \ \simeq \ \int_{\mathbb{R}^3}^{\oplus} H(P) \ d^3P$$

on

$$\mathcal{H} \ \simeq \ \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(P) \ d^3P$$

where

$$\mathcal{H}(P) = \mathcal{A}_N \left[L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes^N} \right] \otimes \mathbb{C}^d \otimes \mathcal{F}$$

for every $P \in \mathbb{R}^3$. The reduced operator $H(P)$ will be explicitly computed and the aim of this article is to initiate the spectral analysis of $H(P)$ when $|P|$ is small. We now introduce the hamiltonian in $\mathcal{H}$ associated to (2.1). As usual we will consider the charge e in front of the quantized electromagnetic fields $A(x)$ and $B(x)$ as a parameter and from now on we will denote it by g. We introduce $\rho(k)$ a cutoff function associated with an ultraviolet cutoff. We suppose that $\rho(k)$ satisfies

$$\int_{|k|\leq 1} \frac{|\rho(k)|^2}{|k|^2} d^3k + \int_{|k|\geq 1} |k||\rho(k)|^2 d^3k < \infty. \tag{2.7}$$

The associated quantized electromagnetic field is then given by $(j = 1, 2, 3)$

$$\begin{aligned} A_j(x, \rho) = \frac{1}{2\pi} \sum_{\mu=1,2} \int d^3k \bigg(& \frac{\rho(k)}{|k|^{1/2}} \epsilon_\mu(k)_j e^{-ik\cdot x} a^\star_\mu(k) \\ & + \frac{\bar{\rho}(k)}{|k|^{1/2}} \epsilon_\mu(k)_j e^{ik\cdot x} a_\mu(k) \bigg) \ , \end{aligned} \tag{2.8}$$

$$B_j(x,\rho) = \frac{i}{2\pi} \sum_{\mu=1,2} \int d^3k \left(-|k|^{1/2}\rho(k) \left(\frac{k}{|k|} \wedge \epsilon_\mu(k) \right)_j e^{-ik\cdot x} a_\mu^\star(k) \right. \tag{2.9}$$
$$\left. + |k|^{1/2}\bar{\rho}(k) \left(\frac{k}{|k|} \wedge \epsilon_\mu(k) \right)_j e^{ik\cdot x} a_\mu(k) \right) .$$

H is then the following operator:

$$\begin{aligned} H = & \sum_{j=1}^{N} \frac{1}{2m} \left(p_j - gA(x_j,\rho)\right)^2 + \frac{1}{2m_{\mathrm{ncl}}} \left(p_{N+1} + ZgA(x_{N+1},\rho)\right)^2 \\ & + V_{cl}(x) \otimes 11 \otimes H_{\mathrm{ph}} - \frac{g}{2m} \sum_{j=1}^{N} \sigma_j \cdot B(x_j) + g_{\mathrm{ncl}} \frac{Zg}{2m_{\mathrm{ncl}}} S \cdot B(x_{N+1}) \end{aligned} \tag{2.10}$$

Let $\mathcal{F}_{0,fin}$ be the set of $(\psi_n)_{n\geq 0} \in \mathcal{F}$ such that ψ_n is in the Schwartz space for every n and $\psi_n = 0$ for all but finitely many n. Then our model is described by the operator H as defined on

$$\mathcal{H}_{0,fin} = \mathcal{A}_N \left[C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)^{\otimes^N} \right] \otimes C_0^\infty(\mathbb{R}^3, \mathbb{C}^d) \otimes \mathcal{F}_{0,fin}.$$

The operator H is then symmetric. In order to compute the reduced hamiltonian $H(P)$ for a given value of the total momentum P we have to introduce the momentum of the center of mass of the electrons the nucleus and, consequently, the Jacobi variables for the atom.

We set

$$r_i = x_{i+1} - \frac{1}{i}(x_1 + x_2 + \cdots + x_i), \quad i = 1, 2, \ldots, N \tag{2.11}$$
$$r_{N+1} = \frac{1}{M}\Big(m(x_1 + x_2 + \cdots + x_N) + m_{\mathrm{ncl}} x_{N+1} \Big), \tag{2.12}$$

where

$$M = Nm + m_{\mathrm{ncl}} \tag{2.13}$$

is the total mass.

Accordingly the corresponding canonically conjugate momentum operators are defined by

$$\omega_j = \frac{1}{i} \frac{\partial}{\partial r_j}, \quad j = 1, 2, \cdots, N+1 \tag{2.14}$$

with

$$\omega_{N+1} = \sum_{j=1}^{N+1} p_j. \tag{2.15}$$

Set

$$X = (x_1, x_2, \ldots, x_{N+1}) \tag{2.16}$$

$$R = (r_1, r_2, \ldots, r_{N+1}). \tag{2.17}$$

We have

$$X = \mathcal{A}R \tag{2.18}$$

where $\mathcal{A}$ is a $(N+1)\times(N+1)$ invertible matrix.

Therefore we have

$$x_j = (\mathcal{A}R)_j, \quad j = 1, 2, \ldots, N+1. \tag{2.19}$$

Similarly we set

$$P_X = (p_1, p_2, \ldots, p_{N+1}) \tag{2.20}$$

$$P_R = (\omega_1, \omega_2, \ldots, \omega_{N+1}) \tag{2.21}$$

and we have

$$P_X = \mathcal{B}P_R \tag{2.22}$$

where $\mathcal{B}$ is a $(N+1)\times(N+1)$ invertible matrix.Notice that in fact, $\mathcal{B}^{-1} = {}^t\mathcal{A}$.

We thus get

$$p_j = (\mathcal{B}P_R)_j, \quad j = 1, 2, \ldots, N+1 \tag{2.23}$$

with

$$\omega_{N+1} = \sum_{l=1}^{N+1} p_l = \left(\mathcal{B}^{-1}P_X\right)_{N+1}.$$

Therefore

$$\mathcal{A}_{j,N+1} = \mathcal{B}^{-1}_{N+1,j} = 1, \quad j = 1, 2, \ldots, N+1. \tag{2.24}$$

Let $\mathcal{M}(x,\rho)$ be the following operator-valued vector

$$\mathcal{M}(x\rho) = (gA(x_1,\rho), \ldots, gA(x_N,\rho), -ZgA(x_{N+1},\rho)) \tag{2.25}$$

i.e.,

$$\begin{aligned}\mathcal{M}(x,\rho)_j &= gA(x_j,\rho), \quad j = 1, 2, \ldots, N\\ \mathcal{M}(x,\rho)_{N+1} &= -ZgA(x_{N+1},\rho).\end{aligned} \tag{2.26}$$

Let $\mu_i > 0$ be the reduced mass defined by

$$\mu_i = \left(\frac{i}{i+1}\right)m, \quad i = 1, 2, \ldots, N. \tag{2.27}$$

We have

$$
\begin{aligned}
&\sum_{j=1}^{N} \frac{1}{2m} \left(p_j - gA(x_j, \rho)\right)^2 + \frac{1}{2m_{\mathrm{ncl}}} \left(p_{N+1} + ZgA(x_{N+1}, \rho)\right)^2 \\
&\qquad = \sum_{j=1}^{N} \frac{1}{2m} \left((\mathcal{B}P_R)_j - gA((\mathcal{A}R)_j, \rho)\right)^2 \\
&\qquad + \frac{1}{2m_{\mathrm{ncl}}} \left((\mathcal{B}P_R)_{N+1} + ZgA((\mathcal{A}R)_{N+1}, \rho)\right)^2 \\
&\qquad = \sum_{j=1}^{N} \frac{1}{2m} \left(\left[\mathcal{B}\left(P_R - \mathcal{B}^{-1}\mathcal{M}(\mathcal{A}R, \rho)\right)\right]_j\right)^2 \\
&\qquad + \frac{1}{2m_{\mathrm{ncl}}} \left(\left[\mathcal{B}\left(P_R - \mathcal{B}^{-1}\mathcal{M}(\mathcal{A}R, \rho)\right)\right]_{N+1}\right)^2 .
\end{aligned}
\tag{2.28}
$$

Now because of the Jacobi coordinates and because, in the Schrödinger representation of Fock space $\mathcal{F}$, $A(x, \rho)$ can be treated as a classical vector potential, we get

$$
\begin{aligned}
&\sum_{j=1}^{N} \frac{1}{2m} \left(p_j - gA(x_j, \rho)\right)^2 + \frac{1}{2m_{\mathrm{ncl}}} \left(p_{N+1} + ZgA(x_{N+1}, \rho)\right)^2 \\
&\qquad = \sum_{j=1}^{N} \frac{1}{2\mu_j} \left(\left(P_R - \mathcal{B}^{-1}\mathcal{M}(\mathcal{A}R, \rho)\right)_j\right)^2 \\
&\qquad + \frac{1}{2M} \left(\left(P_R - \mathcal{B}^{-1}\mathcal{M}(\mathcal{A}R, \rho)\right)_{N+1}\right)^2 .
\end{aligned}
\tag{2.29}
$$

Recall

$$
V_{\mathrm{cl}}(x) = -\sum_{j=1}^{N} \frac{Ze^2}{|x_j - x_{N+1}|} + \sum_{1 \le i < j \le N} \frac{e^2}{|x_i - x_j|}
$$

and set

$$
\widetilde{V}_{\mathrm{cl}}(R) = V_{\mathrm{cl}}(\mathcal{A}R). \tag{2.30}
$$

Note that $\widetilde{V}_{\mathrm{cl}}(R)$ does not depend on r_{N+1}.

By (2.29) H is unitarily equivalent to the following operator defined on

$\mathcal{H}_{0,fin}$ and still denoted by H :

$$\begin{aligned} H &= \frac{1}{2M}\left(\omega_{N+1} - \left(\mathcal{B}^{-1}\mathcal{M}(\mathcal{A}R,\rho)\right)_{N+1}\right)^2 \\ &+ \sum_{j=1}^{N}\frac{1}{2\mu_j}\left(\omega_j - \left(\mathcal{B}^{-1}\mathcal{M}(\mathcal{A}R,\rho)\right)_j\right)^2 \\ &+ \widetilde{V}_{\mathrm{cl}}(R)\otimes 1 + 1\otimes H_{\mathrm{ph}} \\ &- \frac{g}{2m}\sum_{j=1}^{N}\sigma_j\cdot B((\mathcal{A}R)_j) + g_{\mathrm{ncl}}\frac{Zg}{2m_{\mathrm{ncl}}}S\cdot B((\mathcal{A}R)_{N+1}) \end{aligned} \tag{2.31}$$

We now want to show that $H_{N,Z}$ is essentially self adjoint on $\mathcal{H}_{0,fin}$ when g and $\rho(k)$ satisfy appropriate conditions.

Set

$$b_{ij} = (\mathcal{B}^{-1})_{ij}. \tag{2.32}$$

According to (2.24), $b_{N+1,j} = 1$, $j = 1, 2, \ldots, N+1$. We have

$$H = H_0 + H_I \tag{2.33}$$

with

$$H_0 = \left(\frac{\omega_{N+1}^2}{2M} + \sum_{j=1}^{N}\frac{\omega_j^2}{2\mu_j} + \widetilde{V}_{\mathrm{cl}}(R)\right)\otimes 1 + 1\otimes H_{\mathrm{ph}} \tag{2.34}$$

and

$$\begin{aligned} H_I = &-\frac{g}{M}\sum_{j=1}^{N}A((\mathcal{A}R)_i,\rho)\cdot\omega_{N+1} + \frac{Zg}{M}A((\mathcal{A}R)_{N+1},\rho)\cdot\omega_{N+1} \\ &- g\sum_{j=1}^{N}\frac{1}{\mu_j}\left(\sum_{k=1}^{N}b_{jk}A((\mathcal{A}R)_k,\rho)\cdot\omega_j\right) \\ &+ Zg\sum_{j=1}^{N}\frac{1}{\mu_j}b_{j,N+1}A((\mathcal{A}R)_{N+1},\rho)\cdot\omega_j \\ &- \frac{g}{2m}\sum_{j=1}^{N}\sigma_j\cdot B((\mathcal{A}R)_j,\rho) + Zg\frac{g_{\mathrm{ncl}}}{2m_{\mathrm{ncl}}}S\cdot B((\mathcal{A}R)_{N+1},\rho) \\ &+ \frac{1}{2M}\left(\sum_{i=1}^{N}gA((\mathcal{A}R)_i,\rho) - ZgA((\mathcal{A}R)_{N+1},\rho)\right)^2 \\ &+ \sum_{j=1}^{N}\frac{1}{2\mu_j}\left(\sum_{k=1}^{N}b_{jk}gA((\mathcal{A}R)_k,\rho) - b_{j,N+1}ZgA((\mathcal{A}R)_{N+1},\rho)\right)^2 \end{aligned} \tag{2.35}$$

where we used $\varepsilon_\mu(k) \cdot k = 0$.

One checks that

$$\sup_{i,j: i \neq N+1} |b_{ij}| = 1. \tag{2.36}$$

One easily shows that, for $\psi \in \mathcal{A}_N \left[C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)^{\otimes^N} \right] \otimes C_0^\infty(\mathbb{R}^3, \mathbb{C}^d) \otimes \mathcal{F}_{0,fin}$,

$$\begin{aligned} &\frac{|g|}{M} \left\| \left(\sum_{i=1}^N A((\mathcal{A}R)_i, \rho) \cdot \omega_{N+1} \right) \psi \right\| \\ &\leq \frac{12|g|N}{\pi\sqrt{2M}} \left(\int_{\mathbb{R}^3} \frac{|\rho(k)|^2}{|k|^2} d^3k \right)^{\frac{1}{2}} \|(H_0 - E_{elec})\, \psi\| + o(1) \end{aligned} \tag{2.37}$$

where $o(1)$ are terms associated with operators which are relatively bounded with respect to $H_{0,N,Z}$ with a zero relative bound.

Similarly,

$$\begin{aligned} &\frac{Z|g|}{M} \|(A((\mathcal{A}R)_{N+1}, \rho) \cdot \omega_{N+1})\, \psi\| \\ &\leq \frac{12|g|Z}{\pi\sqrt{2M}} \left(\int_{\mathbb{R}^3} \frac{|\rho(k)|^2}{|k|^2} d^3k \right)^{\frac{1}{2}} \|(H_0 - E_{\mathrm{elec}})\, \psi\| + o(1). \end{aligned} \tag{2.38}$$

Here E_{elec} is the infimum of the spectrum of the confining electronic part of $H_{0,N,Z}$, i.e., of

$$\sum_{j=1}^N \frac{\omega_j^2}{2\mu_j} + \widetilde{V}_{\mathrm{cl}}(R).$$

In th same way we verify

$$\begin{aligned} &|g| \left\| \left(\sum_{j=1}^N \frac{1}{\mu_j} \left(\sum_{k=1}^N b_{jk} A((\mathcal{A}R)_k, \rho) \cdot \omega_j \right) \right) \psi \right\| \\ &\leq \frac{12|g|N^2}{\pi\sqrt{2\mu}} \left(\int_{\mathbb{R}^3} \frac{|\rho(k)|^2}{|k|^2} d^3k \right)^{\frac{1}{2}} \|(H_0 - E_{\mathrm{elec}})\, \psi\| + o(1), \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} &Z|g| \left\| \left(\sum_{j=1}^N \frac{1}{\mu_j} b_{j,N+1} A((\mathcal{A}R)_{N+1}, \rho) \cdot \omega_j \right) \psi \right\| \\ &\leq \frac{12|g|ZN}{\pi\sqrt{m}} \left(\int_{\mathbb{R}^3} \frac{|\rho(k)|^2}{|k|^2} d^3k \right)^{\frac{1}{2}} \|(H_0 - E_{\mathrm{elec}})\, \psi\| + o(1), \end{aligned} \tag{2.40}$$

where we used

$$\inf_j \mu_j = \frac{m}{2}.$$

In (2.37), (2.38), (2.39) and (2.40) we have used the following well known estimates:

$$\|a_\mu(g(.,x))\psi\| \leq \left(\int \frac{|g(x,k)|^2}{|k|} d^3k\right)^{1/2} \left\|(I \otimes H_{\rm ph}^{1/2})\psi\right\|$$

and

$$\begin{aligned}\left\|a^*_\mu(g(.,x))\psi\right\| \leq &\left(\int \frac{|g(x,k)|^2}{|k|} d^3k\right)^{1/2} \left\|(I \otimes H_{\rm ph}^{1/2})\psi\right\| \\ &+ \left(\int |g(x,k)|^2 d^3k\right)^{1/2} \|\psi\| \ .\end{aligned} \tag{2.41}$$

Note that

$$\begin{aligned}&\frac{|g|}{2m} \left\|\left(\sum_{j=1}^N \sigma_j \cdot B((\mathcal{A}R)_j, \rho)\right) \psi\right\| = o(1) \\ &Z|g| \frac{g_{\rm ncl}}{2m_{\rm ncl}} \left\|(S \cdot B((\mathcal{A}R)_{N+1}, \rho))\, \psi\right\| = o(1).\end{aligned} \tag{2.42}$$

It remains to estimate the quadratic terms. Let us recall the following estimates (c.f. [3])

$$\begin{aligned}\|a_\mu(f)a_\lambda(f)\psi\| \leq &\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right) \|(H_{\rm ph}+1)\psi\| \\ &+ K \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |\rho(k)|^2 d^3k\right)^{1/2} \left\|(H_{\rm ph}+1)^{1/2}\psi\right\| \ , \\ \left\|a^*_\mu(f)a_\lambda(f)\psi\right\| \leq &\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right) \|(H_{\rm ph}+1)\psi\| \\ &+ \left(K \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |\rho(k)|^2 d^3k\right)^{1/2}\right. \\ &+ \left.\left(\frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\frac{|\rho(k)|^2}{|k|} d^3k\right)^{1/2}\right) \left\|(H_{\rm ph}+1)^{1/2}\psi\right\| \ , \\ \left\|a^*_\mu(f)a^*_\lambda(f)\psi\right\| \leq &\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right) \|(H_{\rm ph}+1)\psi\| \\ &+ \left(K \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |\rho(k)|^2 d^3k\right)^{1/2}\right. \\ &+ \left.\left(\frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\frac{|\rho(k)|^2}{|k|} d^3k\right)^{1/2}\right) \left\|(H_{\rm ph}+1)^{1/2}\psi\right\| \\ &+ \left(\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |k||\rho(k)|^2 d^3k\right)^{1/2} + \int \frac{|\rho(k)|^2}{|k|} d^3k\right) \|\psi\|\end{aligned} \tag{2.43}$$

where $K = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{(1+\lambda)^2} d\lambda$.

As $(H_{\mathrm{ph}}+1)^{\frac{1}{2}}$ is relatively bounded with respect to $H_{\mathrm{ph}}+1$ (and thus to $(H_0 - E_{\mathrm{elec}})$) we deduce that the relative bound of the quadratic terms in $A(\cdot,\rho)$ of (2.35) with respect to $H_0 - E_{\mathrm{elec}}$ is estimated by

$$\left(\frac{2g^2}{M\pi}(N+Z)^2 + \frac{4g^2}{m\pi}(N^3+2ZN^2+Z^2)\right)\int \frac{|\rho(k)|^2}{|k|} d^3k. \tag{2.44}$$

Note that H_0 is essentially self adjoint on $\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right]\otimes$ $C_0^\infty(\mathbb{R}^3,\mathbb{C}^d)\otimes\mathcal{F}_{0,fin}$. Therefore, by (2.37), (2.38), (2.39), (2.40), (2.42) and (2.44), we get the following theorem from the Kato-Rellich theorem.

Theorem 2.1.
Assume (2.7) *and*

$$\begin{aligned}&\frac{6|g|}{\pi}(N+Z)\left(\frac{1}{\sqrt{2M}}+\frac{1}{\sqrt{m}})\right)\left(\int_{\mathbb{R}^3}\frac{|\rho(k)|^2}{|k|}d^3k\right)^{\frac{1}{2}}\\ &+\frac{g^2}{\pi}\left(\frac{(N+Z)^2}{M}+\frac{2(N^2+Z)^2}{m})\right)\left(\int_{\mathbb{R}^3}\frac{|\rho(k)|^2}{|k|}d^3k\right)<\frac{1}{2}.\end{aligned} \tag{2.45}$$

Then H is a self-adjoint operator in $\mathcal{H}$ with domain $D(H)=D(H_0)$ and H is essentially self-adjoint on $\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right]\otimes C_0^\infty(\mathbb{R}^3,\mathbb{C}^d)\otimes\,\mathcal{F}_{0,fin}$.

2.2 The reduced Hamiltonian.

The operator H is invariant by translation. Thus denoting by P the total momentum, i.e., $P=\omega_{N+1}\otimes 1 + 1\otimes \mathrm{d}\Gamma(k)$, H admit a decomposition over the joint spectrum of (P_1,P_2,P_3) as a direct integral

$$H \simeq \int_{\mathbb{R}^3}^{\oplus} H(P)\, d^3P$$

on

$$\mathcal{H} \simeq \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}(P)\, d^3P$$

where

$$\mathcal{H}(P) = \mathcal{A}_N\left[L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}.$$

To compute $H(P)$ we proceed as in [17] and [4]. Let Π be the unitary map from$\mathcal{H}$ to $L^2\left(\mathbb{R}^3, \mathcal{A}_N\left[L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}\right)$ defined by

$$\begin{aligned}&(\Pi\psi)_n\,(r_1,r_2,\dots,r_N,P,(k_1,\mu_1),(k_2,\mu_2),\dots,(k_n,\mu_n))\\ &=\widehat{\psi}_n\left(r_1,r_2,\dots,r_N,P-\sum_{i=1}^n k_i,(k_1,\mu_1),(k_2,\mu_2),\dots,(k_n,\mu_n)\right)\end{aligned} \tag{2.46}$$

where $\widehat{\psi}_n$ is the partial Fourier transformation with respect to r_{N+1}. One easily verifies that, on $\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right]\otimes C_0^\infty(\mathbb{R}^3,\mathbb{C}^d)\otimes\mathcal{F}_{0,fin}$,

$$\begin{aligned}\Pi\omega_{N+1}\Pi^\star &= P-\mathrm{d}\Gamma(k)\\ \Pi A_j\left((\mathcal{A}R)_l,\rho\right)\Pi^\star &= A_j\left((\mathcal{A}\tilde{R})_l,\rho\right)\\ \Pi B_j\left((\mathcal{A}R)_l,\rho\right)\Pi^\star &= B_j\left((\mathcal{A}\tilde{R})_l,\rho\right)\end{aligned} \tag{2.47}$$

where

$$\widetilde{R}=(r_1,r_2,\ldots,r_N,0). \tag{2.48}$$

We remark that $(\mathcal{A}\tilde{R})_k=x_k-r_{N+1}$ for $j=1,\ldots,N+1$ represents the relative postion of the particle k with respect to the center of mass.

Then, for $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^3)\otimes\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^{3N},\mathbb{C}^{2^{\otimes^N}})\right]\otimes\mathbb{C}^d\otimes\mathcal{F}_{0,fin}$ we have

$$\left(\Pi H\Pi^\star\psi\right)(P,\cdot)=H(P)\psi(P,\cdot) \tag{2.49}$$

where the reduced hamiltonian $H(P)$ is given by

$$H(P)=H_0(P)+H_I(P) \tag{2.50}$$

with

$$\begin{aligned}H_0=\sum_{j=1}^N\frac{\omega_j}{2\mu_j}+\widetilde{V}_{\mathrm{cl}}(R)\otimes 1+\\ 1\otimes\left\{\frac{1}{2M}\left(P-\mathrm{d}\Gamma(k)\right)^2+H_{\mathrm{ph}}\right\}\end{aligned} \tag{2.51}$$

and

$$
\begin{aligned}
H_I(P) = & -\frac{g}{M}\sum_{i=1}^{N} A((\mathcal{A}\tilde{R})_i,\rho)\cdot(P-\mathrm{d}\Gamma(k)) \\
& + \frac{Zg}{M}A((\mathcal{A}\tilde{R})_{N+1},\rho)\cdot(P-\mathrm{d}\Gamma(k)) \\
& - g\sum_{j=1}^{N}\frac{1}{\mu_j}\left(\sum_{k=1}^{N} b_{jk}A((\mathcal{A}\tilde{R})_k,\rho)\cdot\omega_j\right) \\
& + Zg\sum_{j=1}^{N} b_{j,N+1}A((\mathcal{A}\tilde{R})_{N+1},\rho)\cdot\omega_j \\
& - \frac{g}{2m}\sum_{j=1}^{N}\sigma_j\cdot B((\mathcal{A}\tilde{R})_j,\rho) + Zg\frac{g_{\mathrm{ncl}}}{2m_{\mathrm{ncl}}}S\cdot B((\mathcal{A}\tilde{R})_{N+1},\rho) \\
& + \frac{1}{2M}\left(\sum_{i=1}^{N} gA((\mathcal{A}\tilde{R})_i,\rho) - ZgA((\mathcal{A}\tilde{R})_{N+1},\rho)\right)^2 \\
& + \sum_{j=1}^{N}\frac{1}{2\mu_j}\left(\sum_{k=1}^{N} b_{jk}gA((\mathcal{A}\tilde{R})_k,\rho) - b_{j,N+1}ZgA((\mathcal{A}\tilde{R})_{N+1},\rho)\right)^2 .
\end{aligned}
\tag{2.52}
$$

For every $P\in\mathbb{R}^3$, $H(P)$ is an operator in $\mathcal{A}_N\left[L^2(\mathbb{R}^{3N},\mathbb{C}^{2^{\otimes N}}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}$. We want to show that this operator has a self-adjoint extension under appropriate conditions on $\rho(\cdot)$ and g.

The operator $\left(\frac{1}{2M}\left(P-\mathrm{d}\Gamma(k)\right)^2+H_{\mathrm{ph}}\right)$ is essentially self-adjoint on $\mathcal{F}_{0,fin}$. Therefore, for every $P\in\mathbb{R}^3$, $H_0(P)$ is essentially self-adjoint on $\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)^{\otimes N}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}_{0,fin}$. $H_0(P)$ still denotes its self-adjoint extension. On the other hand, $H_I(P)$ is a symmetric operator on $\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)^{\otimes N}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}_{0,fin}$ and we want to prove that it is relatively bounded with respect to $H_0(P)$. For that, by (2.39), (2.40), (2.42), (2.44), we only need to estimates the two first terms of the right hand side of (2.52). For $\psi\in\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)^{\otimes N}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}_{0,fin}$, one easily shows that

$$
\begin{aligned}
& \frac{|g|}{M}\sum_{i=1}^{N}\left\|\Big(A((\mathcal{A}\tilde{R})_i,\rho)\cdot(P-\mathrm{d}\Gamma(k))\Big)\psi\right\| \\
& + \frac{Z|g|}{M}\left\|\Big(A((\mathcal{A}\tilde{R})_{N+1},\rho)\cdot(P-\mathrm{d}\Gamma(k))\Big)\psi\right\| \\
& \le \frac{12|g|(N+Z)}{\pi}\left(\frac{1}{\sqrt{2M}}+\frac{1}{\sqrt{2\mu}}\right)\|(H_0(P)-E_{\mathrm{elec}})\psi\| + o(1).
\end{aligned}
\tag{2.53}
$$

Therefore we get

Theorem 2.2.
Assume (2.7) *and* (2.45). *Then, for every* $P \in \mathbb{R}^3$, $H(P)$ *is a self-adjoint operator in* $\mathcal{A}_N\left[L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}$ *with domain* $D(H(P)) = D(H_0(P))$ *and* $H(P)$ *is essentially self-adjoint on* $\mathcal{A}_N\left[C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right]\otimes\mathbb{C}^d\otimes\mathcal{F}_{0,fin}$.

Furthermore we get

Corollary 2.3.
We have

$$\Pi H \Pi^\star = \int_{\mathbb{R}^3}^{\oplus} H(P)\, d^3P$$

The proof of Corollary 2.3 follows by mimicking [4].

3 Main results.

Our main result states that, for $|P|$ and $|g|$ sufficiently small, $H(P)$ has a ground state.

But we have to distinguish the case of atoms, i.e., the case where $Z = N$ from the case of positive ions, i.e., the case where $N < Z$. The problem is connected with the necessity or not of an infrared regularization of the cutoff function in order to prove the existence of a ground state for H. In [2] such an infrared regularization of the cutoff function has been introduced. But, in the case of atoms, we are able to use the Power-Zienau-Woolley transformation (see [20]) in order to get rid of any infrared regularization.

Let h be the following self-adjoint operator in $\mathcal{A}_N\left[L^2(\mathbb{R}^{3N},{\mathbb{C}^2}^{\otimes^N})\right]\otimes\mathbb{C}^d$

$$h = \sum_{j=1}^{N}\frac{\omega_j^2}{2\mu_j} + \widetilde{V}_{\rm cl}(\tilde{R}). \tag{3.1}$$

In order to prove theorems 3.1and 3.2 below we only use the fact that $\inf\sigma(h)$is an isolated eigenvalue of finite multiplicity. Recalling that $V_{\rm cl}(R)$ does not depend on r_{N+1}, we deduce that h is unitarily equivalent to the Zishlin's hamiltonian, $\sum_{j=1}^N \frac{p_j^2}{2m} + V_{\rm cl}(x)$. Therefore it is sufficient to suppose $N \le Z$ (cf. [30]). Notice also that, because of the independence of $V_{\rm cl}(R)$ with respect to r_{N+1}, one has $E_{\rm elec} = \inf\sigma(h)$.

For a bounded below self-adjoint operator T with a ground state, $m(T)$ will denote the multiplicity of $\inf\sigma(T)$.

Our first theorem is concerned with $H_{N,N}$

Theorem 3.1. *(Atoms)*
Assume that $N = Z$ *and that the cutoff function satisfies* (2.7) *and* (2.45) *then there exist* $P_0 > 0$ *and* $g_0 > 0$ *such that, for* $|P| \le P_0$ *and* $|g| \le g_0$, $H(P)$ *has a ground state such that* $m\left(H(P)\right) \le m(h)$.

The second theorem is concerned with positive ions ($N < Z$)

Theorem 3.2. *(Positive ions)*
Assume that $N < Z$ and that the cutoff function satisfies (2.7) (2.45) and

$$\int_{|k|\leq 1} \frac{|\rho(k)|^2}{|k|^3} d^3k < \infty \tag{3.2}$$

Then there exist $\tilde{P}_0 > 0$ and $\tilde{g}_0 > 0$ such that, for $|P| \leq \tilde{P}_0$ and $|g| \leq \tilde{g}_0$, $H(P)$ (with $N < Z$), has a ground state such that $m\,(H(P)) \leq m(h)$.

Remark 3.3.
Theorem 3.1 and theorem 3.2 are still valid for any operator h associated with a potential $V(\tilde{R})$ such that h is essential self-adjoint on $\mathcal{A}_N \left[C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)^{\otimes^N} \right] \otimes \mathbb{C}^d$ and $\inf \sigma(h)$ is an isolated eigenvalue of finite multiplicity.

The proof of these theorems is given in the next section. Notice that the regularization condition (3.2) does not allow $\rho(k) = 1$ near the origin.

A consequence of the existence of a ground state is the existence of asymptotic Fock representations of the CCR.

For $f \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, we define on $D(H_0(P))$ the operators

$$a^\sharp_{\mu,t}(f) := e^{itH(P)} e^{-itH_0(P)} a^\sharp_\mu(f) e^{itH_0(P)} e^{-itH(P)} \ .$$

Let Q be a closed null set such that the polarization vectors $\epsilon_\mu(k)$ are C^∞ on $\mathbb{R}^3 \setminus Q$ for $\mu = 1, 2$. We have

Corollary 3.4.
Suppose that the hypothesis of theorem 3.1 (resp. theorem 3.2) are satisfied. Then, for $f \in C_0^\infty(\mathbb{R}^3 \setminus Q)$ and for every $\Psi \in D(H_0(P))$ the strong limits of $a^\sharp_{\mu,t}(f)$ exist:

$$\lim_{t\to\pm\infty} a^\sharp_{\mu,t}(f)\Psi =: a^\sharp_{\mu,\pm}(f)\Psi \ .$$

The $a^\sharp_{\mu,\pm}$'s satisfy the CCR and, if $\Phi(P)$ is a ground state for $H(P)$, we have for $f \in C_0^\infty(\mathbb{R}^3 \setminus Q)$ and $\mu = 1, 2$

$$a_{\mu,\pm}(f)\Phi(P) = 0 \ .$$

We then deduce the following corollary

Corollary 3.5.
Under the hypothesis of theorem 3.1 (resp. theorem 3.2), the absolutely continuous spectrum of $H(P)$ equals to $[\inf \sigma(H(P)), +\infty)$.

The proofs of these two corollaries follow by mimicking [23, 24].

In what follows we mainly prove theorem 3.1. The proof of theorem 3.2 will then follow easily.

4 Proof of theorem 3.1.

In this section we consider the case of atoms and thus $N = Z$.

To begin with we introduce an infrared regularized cutoff in the interaction Hamiltonian $H_I(P)$. Precisely, for $\sigma > 0$, let ρ_σ be a C_0^∞ regularization of ρ such that

(i) $\rho_\sigma(k) = 0$ for $|k| \leq \sigma$

(ii)

$$\lim_{\sigma\to 0} \int \frac{|\rho_\sigma(k) - \rho(k)|^2}{|k|^j} d^3k = 0, \; j = -1, 1, 2. \tag{4.1}$$

We define $H_{I\sigma}(P)$ as the operator obtained from (2.52) by substituting $\rho_\sigma(k)$ for $\rho(k)$. We then introduce

$$H_\sigma(P) = H_0(P) + H_{I\sigma}(P) \tag{4.2}$$

and we set $E_\sigma(P) := \inf \sigma(H_\sigma(P))$. Theorem 3.1 is a simple consequence of the following result (see [7])

Theorem 4.1.

There exist $g_0 > 0$, $\sigma_0 > 0$ and $P_0 > 0$ such that, for every g satisfying $|g| \leq g_0$, for every σ satisfying $0 < \sigma < \sigma_0$ and for every P satisfying $|P| \leq P_0$, the following properties hold:

(i) For every $\Psi \in D(H_0(P))$ we have $H_\sigma(P)\Psi \to_{\sigma\to 0} H(P)\Psi$

(ii) $H_\sigma(P)$ has a normalized ground state $\Phi_\sigma(P)$ and $E_\sigma(P)$ is an isolated eigenvalue of finite multiplicity of $H_\sigma(P)$.

(iii) Fix $\lambda \in (E_{\mathrm{elec}}, \sigma_{\mathrm{ess}}(h))$. We have

$$\langle \Phi_\sigma(P), P_{(-\infty,\lambda]} \otimes P_{\Omega_{\mathrm{ph}}} \; \Phi_\sigma(P)\rangle \geq 1 - \delta_g(\lambda) \tag{4.3}$$

where $\delta_g(\lambda)$ tends to zero when g tends to zero and $\delta_g(\lambda) < 1$ for $|g| \leq g_0$.

Here $\sigma_{\mathrm{ess}}(h)$ is the essential spectrum of h.

In the last item, $P_{(-\infty,\lambda]}$ is the spectral projection on $(-\infty, \lambda]$ associated to $h(b, V)$ and $P_{\Omega_{\mathrm{ph}}}$ is the orthogonal projection on Ω_{ph}, the vacuum state in $\mathcal{F}$.

Theorem 3.1 is easily deduced from theorem 4.1 as follows. Let $\Phi_\sigma(P)$ be as in theorem 4.1 (ii). Since $\|\Phi_\sigma(P)\| = 1$, there exits a sequence $(\sigma_k)_{k\geq 1}$ converging to zero and such that $(\Phi_{\sigma_k}(P))_{k\geq 1}$ converges weakly to a state $\Phi(P)$. On the other hand, since $P_{(-\infty,\lambda]} \otimes P_{\Omega_{\mathrm{ph}}}$ is finite rank for $\lambda \in (E_{\mathrm{elec}}, \sigma_{\mathrm{ess}}(h))$, it follows from (iii) that for $|g| \leq g_0$ and $|P| \leq P_0$,

$$\langle \Phi(P), P_{(-\infty,\lambda]} \otimes P_{\Omega_{\mathrm{ph}}} \; \Phi(P)\rangle \geq 1 - \delta_g(\lambda)$$

which implies $\Phi(P) \neq 0$. Then we deduce from (i) and from a well known result ([5] lemma 4.9) that $\Phi_N(P)$ is a ground state for $H(P)$.

The result concerning the multiplicity of the ground state is an easy consequence of corollary 3.4 in [24] if g_0 is sufficiently small.

So it remains to prove theorem 4.1. The assertion (i) is easily verified in section 4.1 below. The second assertion is proved in lemma 4.8. Actually the proof of (ii) is lengthy but straightforward since with the infrared cutoff we have a control of the number of photons in term of the energy. The real difficult part is the third one which allows to relax the infrared cutoff. The fundamental lemma in the proof of (iii) is lemma 4.3 which states that, for g and P small enough, the difference $E_\sigma(P-k) - E_\sigma(P)$ is minorized by $-\frac{3}{4}|k|$ uniformly with respect to σ. This estimate is essential to control the number of photons in a ground state of $H_\sigma(P)$ via a pull through formula (see lemma 4.5).

4.1 Proof of (i) of theorem 4.1.

Set $\tilde{\rho}_\sigma = \rho - \rho_\sigma$. We have

$$
\begin{aligned}
&H(P) - H_\sigma(P) = H_I(P) - H_{I\sigma}(P) \\
&= -\frac{g}{M}\sum_{j=1}^{N} A((\mathcal{A}\tilde{R})_i, \tilde{\rho}_\sigma)\cdot(P - \mathrm{d}\Gamma(k)) + \frac{Ng}{M} A((\mathcal{A}\tilde{R})_{N+1}, \tilde{\rho}_\sigma)\cdot(P - \mathrm{d}\Gamma(k)) \\
&- g\sum_{j=1}^{N}\frac{1}{\mu_j}\left(\sum_{k=1}^{N} b_{jk} A((\mathcal{A}\tilde{R})_k, \tilde{\rho}_\sigma)\cdot\omega_j\right) + Ng\sum_{j=1}^{N}\frac{1}{\mu_j} b_{j,N+1} A((\mathcal{A}\tilde{R})_{N+1}, \tilde{\rho}_\sigma)\cdot\omega_j \\
&- \frac{g}{2m}\sum_{j=1}^{N}\sigma_j B((\mathcal{A}\tilde{R})_j, \tilde{\rho}_\sigma) + Ng\frac{g_{\mathrm{ncl}}}{2m_{\mathrm{ncl}}} S\cdot B((\mathcal{A}\tilde{R})_{N+1}, \tilde{\rho}_\sigma) \\
&+ \frac{1}{2M}\left(\sum_{i=1}^{N} gA((\mathcal{A}\tilde{R})_i, \rho) - NgA((\mathcal{A}\tilde{R})_{N+1}, \rho)\right)\cdot \\
&\qquad\qquad\left(\sum_{i=1}^{N} gA((\mathcal{A}\tilde{R})_i, \tilde{\rho}_\sigma) - NgA((\mathcal{A}\tilde{R})_{N+1}, \tilde{\rho}_\sigma)\right) \\
&+ \left(\sum_{i=1}^{N} gA((\mathcal{A}\tilde{R})_i, \tilde{\rho}_\sigma) - NgA((\mathcal{A}\tilde{R})_{N+1}, \tilde{\rho}_\sigma)\right)\cdot \\
&\qquad\qquad\left(\sum_{i=1}^{N} gA((\mathcal{A}\tilde{R})_i, \rho_\sigma) - NgA((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma)\right) \\
&+ \sum_{j=1}^{N}\frac{1}{2\mu_j}\left(\sum_{k=1}^{N} b_{jk} gA((\mathcal{A}\tilde{R})_k, \rho) - b_{j,N+1} NgA((\mathcal{A}\tilde{R})_{N+1}, \rho)\right)\cdot \\
&\qquad\qquad\left(\sum_{k=1}^{N} b_{jk} gA((\mathcal{A}\tilde{R})_k, \tilde{\rho}_\sigma) - b_{j,N+1} NgA((\mathcal{A}\tilde{R})_{N+1}, \tilde{\rho}_\sigma)\right) \\
&+ \left(\sum_{k=1}^{N} b_{jk} gA((\mathcal{A}\tilde{R})_k, \tilde{\rho}_\sigma) - b_{j,N+1} NgA((\mathcal{A}\tilde{R})_{N+1}, \tilde{\rho}_\sigma)\right)\cdot \\
&\qquad\qquad\left(\sum_{k=1}^{N} b_{jk} gA((\mathcal{A}\tilde{R})_k, \rho_\sigma) - b_{j,N+1} NgA((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma)\right).
\end{aligned}
\tag{4.4}
$$

By (4.1) we have $\lim_{\sigma\to 0}\int \frac{|\tilde{\rho}_\sigma(k)|^2}{|k|^j} d^3k = 0$, for $j = -1, 1, 2$. We then deduce from (2.39) – (2.44) , (2.53) and from the Lebesgue's theorem that, for every $\psi \in D(H_0(P))$, $(H(P) - H_\sigma(P))\,\psi \to 0$ as σ tends to zero.

4.2 Main estimates.

In this section we give two lemmas concerning the function $P \mapsto E_\sigma(P)$.

Let $g_1 > 0$ such that (2.45) is satisfied for $|g| \le g_1$.

Lemma 4.2.
There exist $\sigma_0 \in (0,1]$ and a finite constant $C > 0$ which does not depend on $\sigma \in (0, \sigma_0]$ such that

$$E_{\mathrm{elec}} - |g|C \leq E_\sigma(P) \leq E_{\mathrm{elec}} + \frac{P^2}{2M} \tag{4.5}$$

for every $\sigma \in (0, \sigma_0]$, $P \in \mathbb{R}^3$ and $|g| \leq g_1$.

Proof.

Let φ be *a* normalized ground state of h in $\mathcal{A}_N \left[L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes^N} \right] \otimes \mathbb{C}^d$. Since $(a_\mu(k)\Omega_{\mathrm{ph}}, \Omega_{\mathrm{ph}}) = (\Omega_{\mathrm{ph}}, a_\mu(k)^\star \Omega_{\mathrm{ph}}) = 0$ we have

$$\begin{aligned} \left\langle H_(P)\varphi \otimes \Omega_{\mathrm{ph}}, \varphi \otimes \Omega_{\mathrm{ph}} \right\rangle &= \langle H_0(P)\varphi \otimes \Omega_{\mathrm{ph}}, \varphi \otimes \Omega_{\mathrm{ph}} \rangle \\ &= E_{\mathrm{elec}} + \frac{P^2}{2M} \end{aligned}$$

and thus

$$\begin{aligned} E_\sigma(P) &:= \inf \Big\{ (H_\sigma(P)\phi, \phi) \mid \phi \in D(H_0(P)),\ \|\phi\| = 1 \Big\} \\ &\leq E_{\mathrm{elec}} + \frac{P^2}{2M}. \end{aligned} \tag{4.6}$$

On the other hand, let $\widetilde{H}_\sigma$ be the following operator in $\mathcal{A}_N \left[L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes^N} \right] \otimes \mathbb{C}^d \otimes \mathcal{F}$:

$$\widetilde{H}_\sigma = \widetilde{H}_0 + \widetilde{H}_{I\sigma}$$

with

$$\widetilde{H}_0 = h \otimes 1 + 1 \otimes H_{\mathrm{ph}} \tag{4.7}$$

and

$$\begin{aligned} \widetilde{H}_{I\sigma} = &-g \sum_{k=1}^{N} \frac{1}{2\mu_j} \left(\sum_{k=1}^{N} b_{jk} A((\mathcal{A}\tilde{R})_k, \rho_\sigma) \cdot \omega_j + b_{jk} \omega_j \cdot A((\mathcal{A}\tilde{R})_k, \rho_\sigma) \right) \\ &+ Ng \sum_{j=1}^{N} \frac{1}{2\mu_j} \Big(b_{j,N+1} A((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma) \cdot \omega_j + b_{j,N+1} \omega_j \cdot A((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma) \Big) \\ &- \frac{g}{2m} \sum_{j=1}^{N} \sigma_j \cdot B((\mathcal{A}\tilde{R})_j, \rho_\sigma) + Ng \frac{g_{\mathrm{ncl}}}{2m_{\mathrm{ncl}}} S \cdot B((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma) \\ &+ \sum_{j=1}^{N} \frac{1}{2\mu_j} \left(\sum_{k=1}^{N} b_{jk} g A((\mathcal{A}\tilde{R})_k, \rho_\sigma) - b_{j,N+1} N g A((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma) \right)^2 \end{aligned} \tag{4.8}$$

One easily checks that, for $|g| \leq g_1$, $\widetilde{H}_\sigma$ is a self-adjoint operator in $\mathcal{A}_N \left[L^2(\mathbb{R}^{3N}, \mathbb{C}^2)^{\otimes^N} \right] \otimes \mathbb{C}^d \otimes \mathcal{F}$ with domain $D(H_0(P))$. Furthermore, on $D(H_0(P))$, we have

$$H_\sigma(P) = \widetilde{H}_\sigma + \frac{1}{2M}\Big((P - \mathrm{d}\Gamma(k)) - g \sum_{j=1}^{N} A((\mathcal{A}\tilde{R})_j, \rho_\sigma) \\ + NgA((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma) \Big)^2 . \tag{4.9}$$

Hence

$$\inf \sigma(\widetilde{H}_\sigma) \leq E_\sigma(P) \tag{4.10}$$

for every $P \in \mathbb{R}^3$.

By (2.39) – (2.44) and (2.53) which also hold when ρ is replaced by ρ_σ we get that there exist $\sigma_0 \in (0, 1]$ and two constants $b > 0$ and $a > 0$ which do not depend on $\sigma \in (0, \sigma_0]$ and $g \in [-g_1, g_1]$ and which satisfy $bg_1 < 1$ such that,

$$\left\| \widetilde{H}_{I\sigma}\phi \right\| \leq |g| \left(b \left\| \widetilde{H}_0 \phi \right\| + a \left\| \phi \right\| \right) \tag{4.11}$$

for $\phi \in D(\widetilde{H}_0)$ and for $\sigma \in (0, \sigma_0]$.

Therefore, since $\inf \sigma(\widetilde{H}_0) = E_{\mathrm{elec}}$, we obtain, as a consequence of the Kato-Rellich theorem,

$$\inf \sigma(\widetilde{H}_0) \geq E_{\mathrm{elec}} - \max\left(\frac{a|g|}{1 - b|g|}, a|g| + b|g||E_{\mathrm{elec}}| \right). \tag{4.12}$$

We then deduce the lower bound for $E_\sigma(P)$ with

$$C = \max\left(\frac{a}{1 - bg_1}, a + b|E_{\mathrm{elec}}| \right).$$

Lemma 4.3.
There exist $0 < g_2 \leq g_1$ *and* $P_1 > 0$ *such that*

$$E_\sigma(P - k) - E_\sigma(P) \geq -\frac{3}{4}|k| \tag{4.13}$$

uniformly for $k \in \mathbb{R}^3$, $\sigma \in (0, \sigma_0]$, $|g| \leq g_2$ *and* $|P| \leq P_1$.

Remark 4.4.
In this lemma we do not assume that $E_\sigma(P)$ *is an eigenvalue of* $H_\sigma(P)$ *and we will use* (4.13) *in Lemma 4.8 in which we prove that* $H_\sigma(P)$ *has a ground state.*

Proof.

We first remark that, if (4.13) is proved for $H_\sigma(P) + C$ for some constant C, it also holds for $H_\sigma(P)$. Thus, in what follows, we suppose $E_{\mathrm{elec}} = 0$. The proof decomposes in two steps. In the first one, we consider the large values of

$|k|$ (namely $|k| \geq \frac{M}{7}$) while, in the second one, we consider the small values of $|k|$ (namely $|k| \leq \frac{M}{7}$).

From (4.5), we deduce that, uniformly for $\sigma \in (0, \sigma_0]$ and $|g| \leq g_1$, we have for all k and P

$$E_\sigma(P-k) - E_\sigma(P) \geq -\frac{P^2}{2M} - C|g|$$

and thus assuming $|P| \leq \frac{M}{\sqrt{7}}$ and $|g| \leq \frac{M}{28C}$, (4.13) holds true for $|k| \geq \frac{M}{7}$.

Now we suppose $|k| \leq \frac{M}{7}$. As $E_\sigma(P-k)$ belongs to the spectrum of $H_\sigma(P-k)$ there exists a sequence $(\psi_j)_{j\geq 1}$ in $D(H_\sigma(P-k)) = D(H_0(0))$ such that $\|\psi_j\| = 1$ and

$$\lim_{j\to\infty} H_\sigma(P-k)\psi_j - E_\sigma(P-k)\psi_j = 0\ .$$

We then have for every j

$$\begin{aligned}
\langle H_\sigma(P-k)\psi_j, \psi_j\rangle &= \langle H_\sigma(P)\psi_j, \psi_j\rangle + \frac{k^2}{2M} - \frac{k}{M}\cdot\langle (P - \mathrm{d}\Gamma(k))\psi_j, \psi_j\rangle \\
&\quad + \frac{g}{M}\sum_{i=1}^N k\cdot\langle A((\mathcal{A}\tilde{R})_i, \rho_\sigma)\psi_j, \psi_j\rangle \\
&\quad - \frac{Ng}{M} k\cdot\langle A((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma)\psi_j, \psi_j\rangle \\
&\geq E_\sigma(P) + \frac{k^2}{2M} - \left|\frac{k}{M}\cdot\langle (P - \mathrm{d}\Gamma(k))\psi_j, \psi_j\rangle\right| \\
&\quad - \frac{|g|}{M}\left|\sum_{i=1}^N k\cdot\langle A((\mathcal{A}\tilde{R})_i, \rho_\sigma)\psi_j, \psi_j\rangle\right| \\
&\quad - \frac{N|g|}{M}\left|k\cdot\langle A((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma)\psi_j, \psi_j\rangle\right| .
\end{aligned} \tag{4.14}$$

In what follow C will denote any positive constant which does not depend on $P \in \mathbb{R}^3$, $k \in \mathbb{R}^3$, $|g| \leq g_1$, $\sigma \in (0, \sigma_0]$ and $j \geq 1$.

We have

$$\begin{aligned}
\left|\frac{k}{M}\cdot\langle (P - \mathrm{d}\Gamma(k))\psi_j, \psi_j\rangle\right| &\leq \sum_{i=1}^3 \frac{|k_i|}{M}\left|\langle (P_i - \mathrm{d}\Gamma(k_i))\psi_j, \psi_j\rangle\right| \\
&\leq \sum_{i=1}^3 \left(\frac{{k_i}^2}{M} + \frac{|k_i|}{M}\sqrt{2M}\,\|H_0(P-k)\psi_j\|^{1/2}\right) \\
&\leq \frac{|k|^2}{M} + 3|k|\sqrt{\frac{2}{M}}\,\|H_0(P-k)\psi_j\|^{1/2}\ .
\end{aligned} \tag{4.15}$$

On the other hand, by (2.39)-(2.44) and (2.53), one shows that there exists a positive constant $C > 0$ such that

$$\begin{aligned}
&\frac{1}{M}\left|\sum_{i=1}^{N} k \cdot \langle A((\mathcal{A}\tilde{R})_i, \rho_\sigma)\psi_j, \psi_j\rangle\right| \\
&+ \frac{N}{M}\left|k \cdot \langle A((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma)\psi_j, \psi_j\rangle\right| \\
&\leq C|k|(\left\|H_{\mathrm{ph}}^{1/2}\psi_j\right\| + 1) \\
&\leq C|k|(\|H_0(P-k)\psi_j\|^{1/2} + 1) \ .
\end{aligned} \tag{4.16}$$

Now, given $\epsilon > 0$, let J be such that

$$\|H_\sigma(P-k)\psi_j - E_\sigma(P-k)\psi_j\| \leq \epsilon$$

for every $j \geq J$.

Inserting (4.15) and (4.16) in (4.14)

$$\begin{aligned}
E_\sigma(P-k) - E_\sigma(P) \geq -\epsilon - \frac{|k|^2}{2M} - 3|k|\sqrt{\frac{2}{M}}\,\|H_0(P-k)\psi_j\|^{1/2} \\
- |k|C|g|(\|H_0(P-k)\psi_j\|^{1/2} + 1).
\end{aligned} \tag{4.17}$$

It remains to estimate $\|H_0(P-k)\psi_j\|$.

From

$$H_0(P-k)\psi_j = (H_\sigma(P-k) - E_\sigma(P-k))\psi_j + E_\sigma(P-k)\psi_j - H_{I\sigma}(P-k)\psi_j$$

we get for $j \geq J$

$$\|H_0(P-k)\psi_j\| \leq \epsilon + |E_\sigma(P-k)| + \|H_{I\sigma}(P-k)\psi_j\|$$

and we know that there exists a positive constant C such that

$$\|H_{I\sigma}(P)\phi\| \leq |g|C(\|H_0(P)\phi\| + 1) \tag{4.18}$$

for every $P \in \mathbb{R}^3$, $j \geq J$ and $g \in [-g_1, g_1]$.

Thus, choosing $\tilde{g_1} \leq g_1$ such that $\tilde{g_1}C \leq \frac{1}{2}$, we get from (4.17) and (4.18)

$$\|H_0(P-k)\psi_j\| \leq 2\epsilon + 2|E_\sigma(P-k)| + 2|g|C \tag{4.19}$$

for $j \geq J$ and $|g| \leq \tilde{g_1}$. By (4.17) and (4.19) we get

$$\begin{aligned}
E_\sigma(P-k) - E_\sigma(P) \geq -\epsilon - \frac{k^2}{2M} - 6|k|\sqrt{\frac{1}{M}}(\epsilon + |E_\sigma(P-k)| + |g|C)^{1/2} \\
- |k|C|g|((2\epsilon + 2|E_\sigma(P-k)| + 2|g|C)^{1/2} + 1)
\end{aligned}$$

for every $\epsilon > 0$. Hence

$$\begin{aligned}
E_\sigma(P-k) - E_\sigma(P) \geq -|k|\Big(\frac{|k|}{2M} + 6\sqrt{\frac{1}{M}}\,(|E_\sigma(P-k)| + |g|C)^{\frac{1}{2}} \\
+ C|g|\left((2 + 2|E_\sigma(P-k)| + 2|g|C)^{\frac{1}{2}} + 1\right)\Big)
\end{aligned} \tag{4.20}$$

for every $P \in \mathbb{R}^3$ and $k \in \mathbb{R}^3$. From (4.5) we get for $|k| \le \frac{M}{7}$,

$$|E_\sigma(P-k)| \le C|g| + \frac{P^2}{2M} + \frac{|P|}{7} + \frac{M}{98}. \tag{4.21}$$

One then easily shows that there exit $P_2 > 0$ and $g_2 \le \tilde{g_1}$ such that for $|P| \le P_2$, $|k| \le \frac{M}{7}$ and $|g| \le g_2$,

$$E_\sigma(P-k) - E_\sigma(P) \ge -\frac{3}{4}|k| \ .$$

$\square$

4.3 Proof of (iii) of theorem 4.1

In this section we assume that assertion (ii) of theorem 4.1 is already proved (see lemma 4.8). Thus let $\Phi_\sigma(P)$ denote a normalized ground state of $H_\sigma(P)$, i.e.

$$H_\sigma(P)\Phi_\sigma(P) = E_\sigma(P)\Phi_\sigma(P) \ .$$

The main problem in proving (iii) of theorem 4.1 is to control the number of photons in the ground state $\Phi_\sigma(P)$ uniformly with respect to σ. The operator number of photons N_{ph} is given by

$$N_{\mathrm{ph}} = \sum_{j=1,2} \int_{\mathbb{R}^3} d^3k \ a_\mu^\star(k) a_\mu(k).$$

Note that

$$\left\| \left(1 \otimes N_{\mathrm{ph}}^{\frac{1}{2}} \Phi_\sigma(P) \right) \right\|^2 = \sum_{j=1,2} \int_{\mathbb{R}^3} d^3k \ \left\| a_\mu(k)\Phi_\sigma(P) \right\|^2 .$$

We there have the following lemma

Lemma 4.5.
There exists a constant $C > 0$ independent of g and σ such that

$$\|a_\mu(k)\Phi_\sigma(P)\| \le C|g||\rho_\sigma(k)| \left(|k|^{\frac{1}{2}} + \frac{1}{|k|^{\frac{1}{2}}} \right) \left\| (1 + |\tilde{R}|_2)\Phi_\sigma(P) \right\| \tag{4.22}$$

for every $\sigma \in (0, \sigma_0]$, $|g| \le g_2$ and $|P| \le P_2$. Thus

$$\begin{aligned} \left\| (1 \otimes N_{\mathrm{ph}}^{1/2})\Phi_\sigma(P) \right\| \le 4C|g| \left(\int_{\mathbb{R}^3} |\rho_\sigma(k)|^2 \left(\frac{1}{|k|} + |k| \right) d^3k \right)^{1/2} \times \\ \left\| (1 + |\tilde{R}|_2)\Phi_\sigma(P) \right\| \end{aligned} \tag{4.23}$$

where $|\tilde{R}|_2$ is the Euclidian norm of $\tilde{R}$.

Proof.

We use the gauge transformation introduced in [8] and [19]. This transformation is a particular case of the Power-Zienau-Woolley transformation (see [12], [20]). Set

$$\tilde{A}(x,\rho_\sigma) = A(x,\rho_\sigma) - A(0,\rho_\sigma)$$

and

$$U = e^{-ig\sum_{j=1}^{N} r_j\cdot\left(\sum_{l=1}^{N} b_{jl} - N b_{j,N+1}\right)A(0,\rho_\sigma)}. \tag{4.24}$$

We have

$$U a_\mu(k) U^\star = b_\mu(k) = a_\mu(k) + i w_\mu(k,\tilde{R}) \tag{4.25}$$

with

$$w_\mu(k,\tilde{R}) = \frac{g}{2\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}\epsilon_\mu(k)\cdot\sum_{j=1}^{N}\left(\sum_{l=1}^{N} b_{jl} - N b_{j,N+1}\right) r_j$$

In order to estimate $a_\mu(k)\Phi_\sigma(P)$ we write

$$a_\mu(k)\Phi_\sigma(P) = U^\star a_\mu(k)\widetilde{\Phi}_\sigma(P) - i w_\mu(k,\tilde{R})\Phi_\sigma(P)$$

where

$$\widetilde{\Phi}_\sigma(P) = U\Phi_\sigma(P).$$

In order to estimate $\left\| a_\mu(k)\widetilde{\Phi}_\sigma(P)\right\|$ we use the pull through formula. We set

$$\widetilde{H}_\sigma(P) = U H_\sigma(P) U^\star.$$

We have

$$\begin{aligned}
\widetilde{H}_\sigma(P) &= \frac{1}{2M}\left(P - \mathrm{d}\Gamma(k) - g\sum_{i=1}^{N}\tilde{A}\left((\mathcal{A}\tilde{R})_i,\rho_\sigma\right) + gN\tilde{A}\left((\mathcal{A}\tilde{R})_{N+1},\rho_\sigma\right)\right)^2 \\
&+ \sum_{j=1}^{N}\frac{1}{2\mu_j}\left(\omega_j - \sum_{i=1}^{N} g b_{ji}\tilde{A}\left((\mathcal{A}\tilde{R})_i,\rho_\sigma\right) + gN b_{j,N+1}\tilde{A}\left((\mathcal{A}\tilde{R})_{N+1},\rho_\sigma\right)\right)^2 \\
&+ \sum_{\mu=1,2}\int d^3k|k| b_\mu(k)^\star b_\mu(k) + \tilde{V_{\rm el}}(\tilde{R})\otimes 1 \\
&- \frac{g}{2m}\sum_{j=1}^{N}\sigma_j\cdot B\left((\mathcal{A}\tilde{R})_j,\rho_\sigma\right) + g_{\rm ncl}\frac{Ng}{2m_{\rm ncl}} S\cdot B\left((\mathcal{A}\tilde{R})_{N+1},\rho_\sigma\right).
\end{aligned} \tag{4.26}$$

In the first term of the r.h.s. of (4.26) we have used

$$-g\sum_{i=1}^{N} A\left(0,\rho_\sigma\right) + gNA\left(0,\rho_\sigma\right) = 0. \tag{4.27}$$

This is only possible for atoms, i.e., when $N = Z$.

We have

$$\tilde{H}_\sigma(P-k)a_\mu(k)\widetilde{\Phi}_\sigma(P) = a_\mu(k)\tilde{H}_\sigma(P)\widetilde{\Phi}_\sigma(P) - |k|a_\mu(k)\widetilde{\Phi}_\sigma(P) + V_\mu(k)\widetilde{\Phi}_\sigma(P) \tag{4.28}$$

with

$$\begin{aligned}
V_\mu(k) = &-i\frac{g}{2\pi}|k|^{\frac{1}{2}}\rho_\sigma(k)\left(\epsilon_\mu(k)\cdot\sum_{j=1}^{N}\left(\sum_{l=1}^{N}b_{jl} - Nb_{j,N+1}\right)r_j\right) \\
&+ \frac{1}{2M}\frac{g}{2\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}k\cdot\epsilon_\mu(k)\left(\sum_{j=1}^{N}\left(e^{-ik\cdot\tilde{x}_j} - e^{-ik\cdot\tilde{x}_{N+1}}\right)\right) \\
&+ \frac{g}{\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}\epsilon_\mu(k)\cdot\frac{1}{2M}\Bigg(P - k - \mathrm{d}\Gamma(k) - g\sum_{i=1}^{N}\tilde{A}\left(\tilde{x}_i,\rho_\sigma\right) \\
&\qquad\qquad + gN\tilde{A}\left(\tilde{x}_{N+1},\rho_\sigma\right)\Bigg)\left(\sum_{j=1}^{N}\left(e^{-ik\cdot\tilde{x}_j} - e^{-ik\cdot\tilde{x}_{N+1}}\right)\right) \\
&+ \sum_{j=1}^{N}\frac{g}{\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}\epsilon_\mu(k)\cdot\frac{1}{2\mu_j}\times \\
&\qquad\qquad \left(\omega_j - \sum_{l=1}^{N}gb_{jl}\tilde{A}\left(\tilde{x}_l,\rho_\sigma\right) + gN\tilde{A}\left(\tilde{x}_{N+1},\rho_\sigma\right)\right) \\
&\left(\sum_{l=1}^{N}b_{jl}\left(e^{-ik\cdot\tilde{x}_l} - 1\right) - Nb_{j,N+1}\left(e^{-ik\cdot\tilde{x}_{N+1}} - 1\right)\right) \\
&+ \sum_{j=1}^{N}\frac{g}{2\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}\epsilon_\mu(k)\cdot\frac{1}{2\mu_j}\times \\
&\qquad\qquad \left(\sum_{l=1}^{N}b_{jl}k\cdot\frac{\partial x_l}{\partial r_j} - Nb_{j,N+1}k\cdot\frac{\partial x_{N+1}}{\partial r_j}e^{-ik\cdot\tilde{x}_{N+1}}\right) \\
&- \frac{g}{2m}\sum_{j=1}^{N}\sigma_j\cdot|k|^{\frac{1}{2}}\rho_\sigma(k)\left(\frac{k}{|k|}\wedge\epsilon_\mu(k)\right)e^{-ik\cdot\tilde{x}_j} \\
&\qquad\qquad + g_{\mathrm{ncl}}\frac{Ng}{m_{\mathrm{ncl}}}S\cdot|k|^{\frac{1}{2}}\rho_\sigma(k)\left(\frac{k}{|k|}\wedge\epsilon_\mu(k)\right)e^{-ik\cdot\tilde{x}_{N+1}},
\end{aligned} \tag{4.29}$$

where, in order to simplify the notations, we have set

$$\tilde{x}_k = (\mathcal{A}\tilde{R})_k = x_k - r_{N+1}$$

(the last equality is a consequence of (2.24) and shows that $\tilde{x}_k$ represents the relative position of the particule k with respect to the center of mass).

Thus, from (4.28), we get

$$\left(\widetilde{H}_\sigma(P-k) - E_\sigma(P) + |k|\right)a_\mu(k)\widetilde{\Phi}_\sigma = V_\mu(k)\widetilde{\Phi}_\sigma. \tag{4.30}$$

Therefore

$$a_\mu(k)\widetilde{\Phi}_\sigma = R\left(E_\sigma(P-k) - E_\sigma(P) + |k|\right) V_\mu(k)\widetilde{\Phi}_\sigma \tag{4.31}$$

where

$$R(\lambda) = \left(H_\sigma(P-k) - E_\sigma(P-k) + \lambda\right)^{-1} \ . \tag{4.32}$$

In order to estimate $\left\|a_\mu(k)\widetilde{\Phi}_\sigma\right\|$ we have to estimate each term of $R\left(E_\sigma(P-k) - E_\sigma(P) + |k|\right) V_\mu(k)\widetilde{\Phi}_\sigma$. The main terms are those associated with the third and the fourth ones of the r.h.s. of (4.29)

Set

$$\begin{aligned} &T_\mu(k) = \\ &\epsilon_\mu(k)\cdot\frac{1}{2M}\left(P - k - \mathrm{d}\Gamma(k) - g\sum_{i=1}^{N}\tilde{A}\left(\tilde{x}_i,\rho_\sigma\right) + gN\tilde{A}\left(\tilde{x}_{N+1},\rho_\sigma\right)\right). \end{aligned} \tag{4.33}$$

Remark that $T_\mu(k)$ and R are symmetric operator and that $T_\mu(k)R(1)$ is bounded. Thus we write

$$\begin{aligned} &\left\|R\left(E_\sigma(P-k) - E_\sigma(P) + |k|\right)\frac{g}{\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}T_\mu(k)\right. \\ &\qquad\qquad \left.\left(\sum_{j=1}^{N}\left(e^{-ik\cdot\tilde{x}_j} - e^{-ik\cdot\tilde{x}_{N+1}}\right)\right)\widetilde{\Phi}_\sigma\right\| \\ &\le \sup_{\|\varphi\|\le 1}\frac{|g|}{\pi}\frac{|\rho_\sigma(k)|}{|k|^{\frac{1}{2}}}\Big|\Big\langle T_\mu(k)R\left(E_\sigma(P-k) - E_\sigma(P) + |k|\right)\varphi, \\ &\qquad\qquad \sum_{j=1}^{N}\left(e^{-ik\cdot\tilde{x}_j} - e^{-ik\cdot\tilde{x}_{N+1}}\right)\widetilde{\Phi}_\sigma\Big\rangle\Big| \\ &\le \frac{|g|}{\pi}\frac{|\rho_\sigma(k)|}{|k|^{\frac{1}{2}}}\left\|T_\mu(k)R\left(E_\sigma(P-k) - E_\sigma(P) + |k|\right)\right\| \\ &\qquad\qquad \left\|\sum_{j=1}^{N}\left(e^{-ik\cdot\tilde{x}_j} - e^{-ik\cdot\tilde{x}_{N+1}}\right)\widetilde{\Phi}_\sigma\right\|. \end{aligned} \tag{4.34}$$

$T_\mu(k)$ is relatively bounded with respect to $H_0(P-k) - E_{\mathrm{elec}}$ with a relative bound zero. Therefore, for $\epsilon > 0$,

$$\|T_\mu(k)\psi\| \le \epsilon\sqrt{2M}\left\|\left(H_0(P-k) - E_{\mathrm{elec}}\right)\psi\right\| + C_\epsilon\|\psi\| \tag{4.35}$$

for some finite constant $C_\epsilon > 0$ and for $\psi \in D\left(H_0(0)\right)$.

In order to estimate $\left(H_0(P-k) - E_{\mathrm{elec}}\right)\psi$, we write

$$\begin{aligned} \left(H_0(P-k) - E_{\mathrm{elec}}\right)\psi \;&=\; \left(H_\sigma(P-k) - E_\sigma(P) + |k|\right)\psi && (4.36)\\ &+\; \left(E_\sigma(P) - |k|\right)\psi - H_{I\sigma}\psi. && (4.37) \end{aligned}$$

Thus

$$
\begin{aligned}
\|(H_0(P-k)-E_{\text{elec}})\,\psi\| \leq \|(H_\sigma(P-k)-E_\sigma(P)+|k|)\,\psi\| \\
+ (|E_\sigma(P)|+|k|)\,\|\psi\| + \left\|H_{I\sigma(P-k)}\psi\right\|.
\end{aligned}
\tag{4.38}
$$

Note that, by (4.5), there exists a finite constant $M>0$ such that $|E_\sigma(P)| \leq M$ for $\sigma \in (0,\sigma_0]$.

By the theorem 2.2 there exist two finite constants $\alpha>0$ and $\beta>0$ such that $\alpha g_1 < 1$ and

$$
\left\|H_{I\sigma(P-k)}\psi\right\| \leq |g| \left(\alpha \left\|\left(H_0(P-k)-E_{\text{elec}}\right)\psi\right\| + \beta\,\|\psi\|\right)
\tag{4.39}
$$

for $|g| \leq g_1$, $\sigma \in (0,\sigma_0]$ and every P and k.

We get, for some finite constants $C>0$,

$$
\begin{aligned}
\|(H_0(P-k)-E_{\text{elec}})\,\psi\| \leq C\,\|(H_\sigma(P-k)-E_\sigma(P)+|k|)\,\psi\| \\
+ (|k|+1)\,\|\psi\|
\end{aligned}
\tag{4.40}
$$

and by (4.35),

$$
\begin{aligned}
\|T_\mu(k)\psi\| \leq C\,\|(H_\sigma(P-k)-E_\sigma(P)+|k|)\,\psi\| \\
+ (|k|+1)\,\|\psi\|.
\end{aligned}
\tag{4.41}
$$

Now in view of (4.34), we would like to apply (4.41) with $\psi = R\,(E_\sigma(P-k)-E_\sigma(P)+|k|)\,\varphi$. First we remark that, according to Lemma 4.3, we have

$$
E_\sigma(P-k)-E_\sigma(P)+|k| \geq \frac{|k|}{4}
\tag{4.42}
$$

for $\sigma \in (0,\sigma_0]$, $|g| \leq g_2$, $|P| \leq P_2$ and thus we get

$$
\|R\,(E_\sigma(P-k)-E_\sigma(P)+|k|)\| \leq \frac{4}{|k|}\ .
\tag{4.43}
$$

Therefore combining (4.43) and (4.41) we obtain

$$
\|T_\mu(k)R\,(E_\sigma(P-k)-E_\sigma(P)+|k|)\,\varphi\| \leq C\left(1+\frac{1}{|k|}\right)\|\varphi\|
\tag{4.44}
$$

for every $\varphi \in \mathcal{A}_N\left[L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^N}\right] \otimes \mathbb{C}^d \otimes \mathcal{F}$.

Thus

$$
\|T_\mu(k)R\,(E_\sigma(P-k)-E_\sigma(P)+|k|)\| \leq C\left(1+\frac{1}{|k|}\right).
\tag{4.45}
$$

We easily show that

$$\begin{aligned}\left\|\sum_{j=1}^{N}\left(e^{-ik\cdot\tilde{x}_j}-e^{-ik\cdot\tilde{x}_{N+1}}\right)\widetilde{\Phi}_\sigma(P)\right\| &\leq C|k|\left\|(1+|\tilde{x}|_2)\widetilde{\Phi}_\sigma(P)\right\| \\ &\leq C|k|\left\|(1+|\tilde{R}|_2)\widetilde{\Phi}_\sigma(P)\right\|.\end{aligned} \tag{4.46}$$

We then get for the third term of (4.29),

$$\begin{aligned}&\Bigg\| R\left(E_\sigma(P-k)-E_\sigma(P)+|k|\right)\frac{g}{\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}\epsilon_\mu(k)\cdot\frac{1}{2M}\Big(P-k-\mathrm{d}\Gamma(k) \\ &\qquad - g\sum_{i=1}^{N}\tilde{A}\left(\tilde{x}_i,\rho_\sigma\right)+gN\tilde{A}\left(\tilde{x}_{N+1},\rho_\sigma\right)\Big) \\ &\qquad \left(\sum_{j=1}^{N}\left(e^{-ik\cdot\tilde{x}_j}-e^{-ik\cdot\tilde{x}_{N+1}}\right)\right)\widetilde{\Phi}_\sigma\Bigg\| \\ &\leq C\frac{|g|}{\pi}|\rho_\sigma(k)|\left(|k|^{\frac{1}{2}}+\frac{1}{|k|^{\frac{1}{2}}}\right)\left\|(1+|\tilde{R}|_2)\widetilde{\Phi}_\sigma(P)\right\|\end{aligned} \tag{4.47}$$

for $\sigma\in(0,\sigma_0]$, $|g|\leq g_2$, $|P|\leq P_2$.
Similarly, we have for the fourth term of (4.29)

$$\begin{aligned}&\Bigg\| R\left(E_\sigma(P-k)-E_\sigma(P)+|k|\right)\sum_{j=1}^{N}\frac{g}{\pi}\frac{\rho_\sigma(k)}{|k|^{\frac{1}{2}}}\epsilon_\mu(k)\cdot\frac{1}{2\mu_j} \\ &\qquad\times\left(\omega_j-\sum_{l=1}^{N}gb_{jl}\tilde{A}\left(\tilde{x}_l,\rho_\sigma\right)+gN\tilde{A}\left(\tilde{x}_{N+1},\rho_\sigma\right)\right) \\ &\qquad\times\left(\sum_{l=1}^{N}b_{jl}\left(e^{-ik\cdot\tilde{x}_l}-1\right)-Nb_{j,N+1}\left(e^{-ik\cdot\tilde{x}_{N+1}}-1\right)\right)\widetilde{\Phi}_\sigma(P)\Bigg\| \\ &\qquad\qquad\leq C\frac{|g|}{\pi}|\rho_\sigma(k)|\left(|k|^{\frac{1}{2}}+\frac{1}{|k|^{\frac{1}{2}}}\right)\left\|(1+|\tilde{R}|_2)\widetilde{\Phi}_\sigma(P)\right\|.\end{aligned} \tag{4.48}$$

It is easy to verify that the remaining terms of the r.h.s. of, (4.31) associated to the remaining ones of (4.29) are also bounded by

$$C\frac{|g|}{\pi}|\rho_\sigma(k)|\left(|k|^{\frac{1}{2}}+\frac{1}{|k|^{\frac{1}{2}}}\right)\left\|(1+|\tilde{R}|_2)\widetilde{\Phi}_\sigma(P)\right\|.$$

This concludes the proof of the lemma 4.5.

$\square$

Let us remark that the above proof is a little bit formal because of the use of the Pull Through formula. But, by mimicking [24] one easily gets a rigorous proof. We omit the details.

The following Lemma allows us to control $\left\|(1+|\tilde{R}|_2)\widetilde{\Phi}_\sigma(P)\right\|$. Let $\delta = \mathrm{dist}(E_{\mathrm{elec}}, \sigma(h)\backslash\mathcal{E}_{\mathrm{elec}} > 0\}$. By (4.5), there exists $P_3 > 0$ and $0 < g_3 \leq g_2$ such that

$$\begin{aligned} E_\sigma(P) &\leq E_{\mathrm{elec}} + \frac{\delta}{3}, \text{ for } |P| \leq P_3 \text{ and for } \sigma \in (0, \sigma_0] \\ C|g| &\leq \frac{\delta}{12}, \text{ for } |g| \leq g_3 \end{aligned} \tag{4.49}$$

where C is the constant in (4.5).

Let Δ be an interval such that $E_\sigma(P) \in \Delta$ for $|P| \leq P_3$ and for $\sigma \in (0, \sigma_0]$ and $\sup\Delta < E_{\mathrm{elec}} + \frac{\delta}{2}$.

Thus

$$E_{\mathrm{elec}} + \frac{2\delta}{3} - \sup\Delta - C|g| \geq \frac{\delta}{12} \tag{4.50}$$

for $|P| \leq P_3$ and $|g| \leq g_3$

Let $\eta > 0$ be such that

$$0 < \eta^2 < E_{\mathrm{elec}} + \frac{2\delta}{3} - \sup\Delta - C|g| \ . \tag{4.51}$$

We then have

Lemma 4.6.
There exists a finite constant $M_\Delta > 0$ such that

$$\left\|(e^{\eta|\tilde{R}|_2} \otimes 1)\Phi_\sigma(P)\right\| \leq M_\Delta \tag{4.52}$$

for $|P| \leq P_3$, $|g| \leq g_3$ and $\sigma \in (0, \sigma_0]$.

The proof of lemma 4.6 easily follows by mimiking the proof of theorem II.1 in [7].

We denote by $P_{(.]}$ the spectral measure of h and by $P_{\Omega_{\mathrm{ph}}}$ the orthogonal projection on Ω_{ph}. We have the following lemma

Lemma 4.7.
Fix $\lambda \in (E_{\mathrm{elec}}, \inf\sigma_{\mathrm{ess}}(h))$. There exists $\delta_g(\lambda) > 0$ such that $\delta_g(\lambda) \to 0$ when $g \to 0$ and

$$\langle P_{[\lambda,\infty)} \otimes P_{\Omega_{\mathrm{ph}}}\Phi_\sigma(P) \ , \ \Phi_\sigma(P)\rangle \leq \delta_g(\lambda) \tag{4.53}$$

for every $\sigma \in (0, \sigma_0]$, $|P| \leq P_3$ and $|g| \leq g_3$.

Proof. Since $P_{\Omega_{\mathrm{ph}}} H_{\mathrm{ph}} = 0$ and $P_{\Omega_{\mathrm{ph}}}(P - \mathrm{d}\Gamma(k))^2 = P^2 P_{\Omega_{\mathrm{ph}}}$ we get

$$\begin{aligned} &(P_{[\lambda,\infty)} \otimes P_{\Omega_{\mathrm{ph}}})(H_\sigma(P) - E_\sigma(P)) = P_{[\lambda,\infty)}(h \otimes I) \otimes P_{\Omega_{\mathrm{ph}}} \\ &\quad + (\frac{P^2}{2M} - E_\sigma(P))P_{[\lambda,\infty)} \otimes P_{\Omega_{\mathrm{ph}}} + P_{[\lambda,\infty)} \otimes P_{\Omega_{\mathrm{ph}}} H_{I\sigma}(P) \ . \end{aligned} \tag{4.54}$$

Applying this last equality to $\Phi_\sigma(P)$ we get

$$\begin{aligned} 0 &= P_{[\lambda,\infty)}(h\otimes I)\otimes P_{\Omega_{\rm ph}}\Phi_\sigma(P) \\ &\quad + (\frac{P^2}{2M} - E_\sigma(P))P_{[\lambda,\infty)}\otimes P_{\Omega_{\rm ph}}\Phi_\sigma(P) \\ &\quad + P_{[\lambda,\infty)}\otimes P_{\Omega_{\rm ph}}H_{I\sigma}(P)\Phi_\sigma(P)\ . \end{aligned} \tag{4.55}$$

Since $hP_{[\lambda,\infty)} \geq \lambda P_{[\lambda,\infty)}$ we obtain from lemma 4.3 and (4.4)

$$\begin{aligned} &\langle P_{[\lambda,\infty)}\otimes P_{\Omega_{\rm ph}}\Phi_\sigma(P)\ ,\ \Phi_\sigma(P)\rangle \leq \\ &\qquad \frac{1}{E_{\rm elec}-\lambda}\langle (P_{[\lambda,\infty)}\otimes P_{\Omega_{\rm ph}})H_{I\sigma}(P)\Phi_\sigma(P)\ ,\ \Phi_\sigma(P)\rangle \end{aligned} \tag{4.56}$$

for every $\sigma\in(0,\sigma_0]$, $|P|\leq P_3$ and $|g|\leq g_3$. The lemma then follows from (4.56) and (4.39).

□

We are now able to conclude the proof of (iii) of theorem 4.1. We have

$$\begin{aligned} &\langle P_{(-\infty,\lambda]}\otimes P_{\Omega_{\rm ph}}\Phi_\sigma(P)\ ,\ \Phi_\sigma(P)\rangle = \\ &\qquad\qquad 1 - \langle P_{[\lambda,\infty)}\otimes P_{\Omega_{\rm ph}}\Phi_\sigma(P)\ ,\ \Phi_\sigma(P)\rangle \\ &\qquad\qquad - \langle 1\otimes P^\perp_{\Omega_{\rm ph}}\Phi_\sigma(P)\ ,\ \Phi_\sigma(P)\rangle\ \ . \end{aligned} \tag{4.57}$$

The second term in the r.h.s. of (4.57) is estimated by lemma 4.7 and, noticing that $P^\perp_{\Omega_{\rm ph}}\leq N_{\rm ph}$, the two last terms are estimated by lemma 4.5 and lemma 4.6. Theorem 3.1 then follows by choosing $P_0=\inf(P_1,P_2,P_3)$ and $g_0=\inf(g_1,g_2,g_3)$ and from the following Lemma

□

Lemma 4.8.
$H_\sigma(P)$ has a ground state for $0<\sigma\leq\sigma_0$, $|P|\leq P_0$ and $|g|\leq g_0$.

In this lemma we prove the assertion (ii) of theorem 4.1 : for σ and P small enough, the Hamiltonian with infrared cutoff has a ground state. This result is not surprising but the complete proof is long. Actually it follows by mimicking [16, 17, 13] (see also [2] and [27]) and, here, we only sketch the proof.

First we are faced with the lack of smoothness of the $\epsilon_\mu(k)$'s which define vector fields on spheres $|k|=$cst (see [26, 18]). It suffices to consider one example. From now on suppose that

$$\epsilon_1(k) = \frac{1}{\sqrt{k_1^2+k_2^2}}(k_2,-k_1,0) \quad\text{and}\quad \epsilon_2(k) = \frac{k}{|k|}\wedge\epsilon_1(k)\ .$$

The functions $\epsilon_\mu(k)$, $\mu=1,2$, are smooth only on $\mathbb{R}^3\setminus\{(0,0,k_3)\mid k_3\in\mathbb{R}\}$. Nevertheless, in our case, we can overcome this problem easily by choosing the regularization ρ_σ of ρ as a C^∞ function whose support does not intersect the line $\{(0,0,k_3)\mid k_3\in\mathbb{R}\}$. From now on we suppose that it is the case.

Let $\omega_{\rm mod}(k)$ be the modified dispersion relation as defined in ([17], section 5, hypothesis 3), i.e. : $\omega_{\rm mod}(k)$ is a smooth function satisfying

(i) $\omega_{\mathrm{mod}}(k) \geq \max(|k|, \frac{\sigma}{2})$ for all $k \in \mathbb{R}^3$, $\omega_{\mathrm{mod}}(k) = |k|$ for $|k| \geq \sigma$.

(ii) $|\nabla \omega_{\mathrm{mod}}(k)| \leq 1$ for all $k \in \mathbb{R}^3$, and $\nabla \omega_{\mathrm{mod}}(k) \neq 0$ unless $k = 0$.

(iii) $\omega_{\mathrm{mod}}(k_1 + k_2) \leq \omega_{\mathrm{mod}}(k_1) + \omega_{\mathrm{mod}}(k_2)$ for all $k_1, k_2 \in \mathbb{R}^3$.

We set

$$H_{ph,\mathrm{mod}} = \sum_{\mu=1,2} \int \omega_{\mathrm{mod}}(k) a^{\star}_{\mu}(k) a_{\mu}(k) d^3k$$

and let $H_{\mathrm{mod},\sigma}(P)$ be the same Hamiltonian as in (2.50) except that in (2.51) we replace H_{ph} by $H_{ph,mod}$.

Theorem 2.2, with the same assumption (2.45), is still valid for $H_{\mathrm{mod},\sigma}(P)$. Set $E_{\mathrm{mod},\sigma}(P) := \inf \sigma(H_{\mathrm{mod},\sigma}(P))$. Then $E_{\mathrm{mod},\sigma}(P)$ still satisfies lemma 4.3 and (4.13) for the same constants g_2 and P_1. Moreover, according to ([17]; thm 3), $E_\sigma(P) = E_{\mathrm{mod},\sigma}(P)$ for $|P| \leq P_1$ and $|g| \leq g_2$ and $E_\sigma(P)$ is an eigenvalue of $H_\sigma(P)$ if and only if $E_{\mathrm{mod},\sigma}(P)$ is an eigenvalue of $H_{\mathrm{mod},\sigma}(P)$. Thus in order to prove that $H_\sigma(P)$ has a ground state it suffices to prove that $E_{\mathrm{mod},\sigma}(P) < \inf \sigma_{\mathrm{ess}}(H_{\mathrm{mod},\sigma}(P))$. The proof is by contradiction.

It follows by proving that, if $E_{\mathrm{mod},\sigma}(P) = \inf \sigma_{\mathrm{ess}}(H_{\mathrm{mod},\sigma}(P))$, then we can construct a sequence of states (ϕ_n) whose energy converges to $E_{\mathrm{mod},\sigma}(P)$ and that have a non vanishing component along the delocalized photons and therefore has an energy which is larger than $E_{\mathrm{mod},\sigma}(P) + \frac{\sigma}{2}$ (c.f. [17] or [2] for details).

5 Proof of theorem 3.2.

When $N < Z$ the above proof only fails in lemma 5.3 where we used $N = Z$ in order to obtain (4.27) and thus (4.26).

We now haveThis implies that, now, instead of (4.32), we have to consider

$$\begin{aligned} T_\mu(k) =& \epsilon_\mu(k) \cdot \frac{1}{2M} \times \\ & \left(P - k - \mathrm{d}\Gamma(k) - g \sum_{i=1}^{N} A\left((\mathcal{A}\tilde{R})_i, \rho_\sigma \right) + gZA\left((\mathcal{A}\tilde{R})_{N+1}, \rho_\sigma \right) \right). \end{aligned} \tag{5.1}$$

We cannot cancel the singularity $\frac{1}{|k|^{\frac{1}{2}}}$ in $A(\cdot, \rho_\sigma)$ as we have done when $N = Z$ by substituting $\tilde{A}(\cdot, \rho_\sigma)$ for $A(\cdot, \rho_\sigma)$ and by applying the unitary transformation U now given by

$$U = e^{-ig \sum_{j=1}^{N} r_j \cdot \left(\sum_{k=1}^{N} b_{jk} - Z b_{j,N+1} \right) A(0,\rho_\sigma)}. \tag{5.2}$$

Therefore, in order to estimate all the terms associated with $T_\mu(k)$, we have to suppose that

$$\int_{|k| \leq 1} \frac{|\rho_\sigma(k)|^2}{|k|^3} d^3k < \infty \tag{5.3}$$

for $\sigma \in (0, \sigma_0]$.

Thus we get

$$\left\|(1 \otimes N_{\mathrm{ph}}^{1/2})\Phi_\sigma(P)\right\| \leq C|g| \left(\int_{\mathbb{R}^3} |\rho_\sigma(k)|^2 \left(\frac{1}{|k|^3} + |k|\right) d^3k\right)^{1/2} \|\Phi_\sigma(P)\| \quad (5.4)$$

for $\sigma \in (0, \sigma_0]$, $|g| \leq g_2$ and $|P| \leq P_2$. We then conclude the proof of theorem 3.2 as above by choosing by choosing $P_0 = \inf(P_1, P_2)$ and $g_0 = \inf(g_1, g_2)$.

□

6 The hydrogen atom in a constant magnetic field.

We consider a hydrogen atom in $\mathbb{R}^3$ interacting with a classical magnetic field B_0 pointing along the x_3-axis and with a quantized electromagnetic field.

The index 1 is related with the electron while the index 2 is related with the proton. So we introduce in the Hilbert space

$$\mathcal{H}_{\mathrm{magn}} = L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

The hamiltonian is the following one:

$$\begin{aligned} H_{\mathrm{magn}} = &\frac{1}{2m_1}\left(p_1 - \frac{e}{2}B_0 \wedge x_1 - gA(x_1, \rho)\right)^2 \\ &+ \frac{1}{2m_2}\left(p_2 + \frac{e}{2}B_0 \wedge x_2 + gA(x_2, \rho)\right)^2 \\ &+ V(x_1 - x_2) \otimes 1 + 1 \otimes H_{\mathrm{ph}}. \end{aligned}$$

Here for simplicity, we omit spins and we still replace the charge e in front of $A(\cdot, \rho)$ by the parameter g. V denote the coulomb potential: $V(x) = -e^2\backslash|x$.

The total momentum of the system is given by $P = K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k)$ where

$$K = p_1 + \frac{e}{2}B_0 \wedge x_1 + p_2 - \frac{e}{2}B_0 \wedge x_2.$$

One easily verifies that the 3 components of P commute with H_{magn} and that $[P_j, P_k] = 0$ for $j, k = 1, 2, 3$. Therefore H_{magn} admits a decomposition as a direct integral over the joint spectrum of (P_1, P_2, P_3).

Remark 6.1.
In fact $K \wedge K = -iB_0(e_1 + e_2)$ and the three components of P commutes together only when the total charge is zero. This explain the difference with the case of the electron that we considered in [2] for which we were able to reduce the hamiltonian only with respect to the third component of the total momentum.

In order to obtain a simpler reduced hamiltonian we prefer first to transform H_{magn} in such a way the momentum K is transformed into P_R, the conjugate momentum of the center of mass R.

Considering the unitary transformation

$$U = e^{i\frac{e}{2}r\cdot B_0\wedge R}$$

with $r = x_1 - x_2$, one obtains

$$U^{-1}KU = P_R$$

and denoting $\widetilde{H}_{\mathrm{magn}} = U^{-1}H_{\mathrm{magn}}U$ one gets

$$\begin{aligned}\widetilde{H}_{\mathrm{magn}} &= \frac{1}{2m_1}\left(\frac{m_1}{M}P_R + p_r - \frac{e}{2}B_0\wedge r - gA(x_1,\rho)\right)^2\\ &+ \frac{1}{2m_2}\left(\frac{m_2}{M}P_R - p_r - \frac{e}{2}B_0\wedge r + gA(x_2,\rho)\right)^2\\ &+ V(r)\otimes 1 + 1\otimes H_{\mathrm{ph}}\end{aligned}$$

where $p_r = -i\nabla_r$.

We have

$$[P_R, \widetilde{H}_{\mathrm{magn}}] = 0.$$

Therefore the hamiltonian $\widetilde{H}_{\mathrm{magn}}$, unitarily equivalent to H_{magn}, admits a decomposition as a direct integral over the joint spectrum of the three components of $\widetilde{P} = P_R\otimes 1 + 1\otimes \mathrm{d}\Gamma(k)$. We obtain formally

$$\widetilde{H}_{\mathrm{magn}} \;\simeq\; \int_{\mathbb{R}^3}^{\oplus} \widetilde{H}_{\mathrm{magn}}(P)\; d^3P$$

on

$$\mathcal{H}_{\mathrm{magn}} \;\simeq\; \int_{\mathbb{R}^3}^{\oplus} L^2(\mathbb{R}^3)\otimes\mathcal{F}\; d^3P$$

with

$$\begin{aligned}\widetilde{H}_{\mathrm{magn}}(P) &= \frac{1}{2m_1}\left(\frac{m_1}{M}\left(P - \mathrm{d}\Gamma(k)\right) + p - \frac{e}{2}B_0\wedge r - eA\left(\frac{m_2}{M}r,\rho\right)\right)^2\\ &+ \frac{1}{2m_2}\left(\frac{m_2}{M}\left(P - \mathrm{d}\Gamma(k)\right) - p - \frac{e}{2}B_0\wedge r + eA\left(-\frac{m_1}{M}r,\rho\right)\right)^2\\ &- \frac{e^2}{r}\otimes 1 + 1\otimes H_{\mathrm{ph}},\end{aligned}$$

where $p = p_r$.

Precisely, as usual we replace the charge e by a parameter g in the interaction terms and write

$$\widetilde{H}_{\mathrm{magn}}(P) = \widetilde{H}_{\mathrm{Hyd},0}(P) + \widetilde{H}_{\mathrm{magn},I}(P)$$

with, introducing the electronic hamiltonian

$$h(B_0) = \frac{1}{2m_1}\left(p - \frac{e}{2}B_0\wedge r\right)^2 + \frac{1}{2m_2}\left(p + \frac{e}{2}B_0\wedge r\right)^2 - \frac{e^2}{r}\ ,$$

one has

$$H_{\mathrm{magn},0}(P) = h(B_0) \otimes 1 + \frac{1}{2M} \left(P - \mathrm{d}\Gamma(k)\right)^2 + 1 \otimes H_{\mathrm{ph}}.$$

and

$$\begin{aligned}\widetilde{H}_{\mathrm{magn},I}(P) = & \frac{g}{M} \left(P - \mathrm{d}\Gamma(k)\right) \left(A\left(-\frac{m_1}{M} r, \rho\right) - A\left(\frac{m_2}{M} r, \rho\right)\right) \\ & + \frac{g^2}{2m_1} A\left(\frac{m_2}{M} r, \rho\right)^2 + \frac{g^2}{2m_2} A\left(-\frac{m_1}{M} r, \rho\right)^2 \\ & - \frac{g}{M} \left(P - \mathrm{d}\Gamma(k)\right) B_0 \wedge r \ .\end{aligned}$$

From [6] we know that the electronic hamiltonian $h(B_0)$ has an isolated ground state φ_{el}. Hence, for $|P| < M$, $\varphi_{\mathrm{el}} \otimes \Omega_{\mathrm{ph}}$ is the ground state of $\widetilde{H}_{\mathrm{magn},0}(P)$ and remains isolated. Then following the same steps as in section 4 with straightforward adaptation of lemmas 4.2, 4.3 and 4.5 one obtains the following theorem

Theorem 6.2.
Assume that the cutoff function satisfies (2.7). *Then there exist* P_0 *and* $g_0 > 0$ *such that for* $|P| \leq P_0$ *and* $|g| \leq g_0$, $\widetilde{H}_{\mathrm{magn}}(P)$ *has a ground state with*

$$m\left(\widetilde{H}_{\mathrm{magn}}(P)\right) \leq m\left(h(B_0)\right)) \, .$$

□

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